Supplement to "Does Risk Shape Economies? Income Volatility and Structural Change"

ONLINE APPENDIX

Fenicia Cossu[1] Alessio Moro[2] Andrea Mottola[3]

1 Data

1.1 Figure 1

Figure 1 of the main text is constructed as follows. For the cross-country sample, we use Latin American and Asian economies from the GGDC 10-Sector Database and Penn World Table 10.01 over the period 1950–2010. In the top panel, the dependent variable is the services share in GDP, while the horizontal axis is the log real GDP per capita. We estimate a panel regression (using all Latin American, Asian countries and the U.S.) of the services share on a cubic polynomial in log GDP per capita, that includes country fixed effects. The black fitted line reports the predicted relationship from this regression (including fixed effects), while the colored dots report country-year observations net of country fixed effects. The historical U.S. series is constructed from Herrendorf, Rogerson, and Valentinyi (2014) and plotted in the same space by normalizing it such that it coincides with the modern U.S. series in 2008. The orange line then reports an estimated relationship between *these* U.S. historical services shares, which live on the figure's vertical axis, on a cubic polynomial in log GDP per capita. The bottom panel repeats the same procedure using GDP volatility as the dependent variable. GDP volatility is computed as the standard deviation of percentage deviations of real GDP per capita from an HP-filtered trend over the previous 15 years[4]. This construction allows us to compare, at similar levels of GDP per capita, both the services share and the level of macroeconomic risk faced by the historical U.S. and by late industrializers in Latin America and Asia.

[1]University of Cagliari. E-mail: fenicia.cossu@unica.it.
[2]University of Cagliari, CEPR and IZA@LISER. E-mail: amoro@unica.it.
[3]University of Cagliari. E-mail: andrea.mottola@unica.it.
[4]See Section 1.5 in this supplement

1.2 U.S. Historical Data from 1900 to 2008

We rely on the historical time series compiled by Herrendorf, Rogerson, and Valentinyi (2014), which integrate data from Carter, Gartner, Haines, Olmstead, Sutch, Wright, et al. (2006) for the pre-1930 period with data from the Bureau of Economic Analysis (BEA) starting in 1929. This dataset is further supplemented with GDP per capita estimates from Maddison (2010), covering the years 1800–2000. The dataset provides historical U.S. data on sectoral value added (for agriculture, manufacturing, and services) and personal consumption expenditures, both expressed in millions of U.S. dollars at current prices. Although the GDP per capita series begins in 1800, sectoral value added data, necessary for computing sectoral shares, are available on a continuous yearly basis only from 1909 onward, while personal consumption expenditure data are available starting in 1900. Given that our volatility indicator requires a 15-year backward window at each year t, we use Maddison's real GDP per capita series (expressed in 1990 international Geary-Khamis dollars) beginning in 1894. Furthermore, we compute the sectoral shares - Agriculture, Manufacturing, and Services on both the value added and consumption sides, defined as the ratio of each sector's value to the total of the three sectors. The value added shares cover the period from 1909 to 2008, while the consumption shares span the years 1900 to 2008.

1.3 GGDC 10-Sector Database

The GGDC 10-Sector Database, provided by the Groningen Growth and Development Centre (GGDC), provides annual data for 42 countries across Africa, Asia, Europe, Latin America, the Middle East, and North America. Covering the period from 1947 to 2013, the database includes information on persons employed, gross value added at current national prices, and gross value added at constant 2005 national prices for 10 broad economic sectors. For the purpose of our analysis, we aggregate the 10 sectors into three broad categories using data from the period 1950 to 2010. Agriculture corresponds to 'Agriculture, hunting, forestry, and fishing' (AtB). Manufacturing includes 'Mining and quarrying' (C), 'Manufacturing' (D), 'Electricity, gas, and water supply' (E), and 'Construction' (F). Services encompass 'Wholesale and retail trade, hotels, and restaurants' (GtH), 'Transport, storage, and communication' (I), 'Finance, insurance, real estate, and business services' (JtK), and 'Community, social, and personal services' (OtP) and 'Government services' (LtN).

We compute sectoral shares - Agriculture, Manufacturing, and Services - as the ratio of each aggregated sector's nominal value added to the total of the three aggregated sectors. In the cross-country regressions we use the following economies. For ASIA: China (CHN, 1952 to 2010), Indonesia (IDN, 1960 to 2010), India (IND, 1950 to 2010), Korea Republic (KOR, 1953 to 2010), Malaysia (MYS, 1970 to 2010), Philippines (PHL, 1971 to 2010), Thailand (THA, 1951 to 2010), Taiwan (1951 to 2010). For LATIN AMERICA: Argentina (ARG, 1950 to 2010), Bolivia (BOL, 1958 to 2010), Brazil (BRA, 1990 to 2010), Chile (CHL, 1950 to 2010), Colombia (COL, 1950 to 2010), Costa Rica (CRI, 1950 to 2010), Mexico (MEX, 1950 to 2010). We exclude Peru (PER) and Venezuela (VEN) from the Latin American sample. Peru reports negative value-added shares for some sectors in several years. Venezuela is an oil-driven economy whose industrial share largely reflects hydrocarbon rents rather than manufacturing activity, producing a structural-transformation trajectory not comparable with that of the other countries in the sample. We acknowledge, however, that most results for Latin America hold with the inclusion of Venezuela.

1.4 GGDC PWT 10.01

We use the Penn World Table (PWT) version 10.01, provided by the Groningen Growth and Development Centre (GGDC). For our analysis, we construct a GDP per capita series by dividing the "Real GDP at constant 2017 national prices - RGDPNA" (in millions of 2017 U.S. dollars) by "Population" (in millions).

1.5 Volatility Indicator based on GDP and Aggregate Consumption

To construct our volatility indicator used in Section 4 of the main text, we apply the Hodrick–Prescott (HP) filter to the logarithmic transformation of the real GDP per capita series in order to isolate its cyclical component. The smoothing parameter is set to $\lambda = 6.25$, following Ravn and Uhlig (2002) for annual data. We compute a rolling standard deviation of the log-deviations from trend over the previous 15 years - from $t-15$ to $t-1$. This is our measure of volatility $\sigma_{t:-15}$ at time t that we use in Tables 1 and 6 of the main text.

1.6 Geopolitical Risk index

As an alternative proxy for macroeconomic uncertainty, we use the Historical Geopolitical Risk (GPR) Index developed by Caldara and Iacoviello (2022). This index quantifies the intensity of geopolitical tensions using a news-based methodology that measures the frequency of articles discussing wars, terrorism, and international crises in major newspapers (e.g., New York Times, Washington Post, Chicago Tribune). Their historical index is constructed from 1900 onward at monthly frequency, based on the share of articles containing pre-specified keywords related to geopolitical threats and acts. To integrate this indicator into our annual U.S. database, we compute, for each year, the simple average of the twelve monthly historical GPR values, thus generating a yearly series of geopolitical risk consistent with the temporal resolution of our main dataset.

1.7 Cross-sectional data for the U.S.

Our empirical analysis relies on data from the U.S. Consumer Expenditure Survey (CEX), which provides detailed information on household consumption, income, and demographic characteristics. We use the quarterly consumer unit files covering the period 2014–2019, focusing on cross-sectional variation in individual spending behavior.

The dependent variable, the share of services in total consumption, is constructed following the methodology proposed by Boppart (2014). In particular, we define service consumption to include the following categories: food away from home; shelter; utilities, fuels, and public services; other vehicle expenses; public transportation; health care; personal care; education; cash contributions; personal insurance; and pensions. The dataset also includes a wide range of individual-level control variables, such as age, household size, and geographical identifiers.

To measure occupational-level income risk, we link CEX respondents to a set of external indicators based on their reported occupation. These indicators capture different dimensions of labor market uncertainty and include both technology-related proxies, such as direct measures of employment instability - as the duration of unemployment spells and exposure to automation and routinization.

Among the indicators used to proxy occupational income risk, $risk^{(1)}_{i,j,t}$ is computed by OCC1990 occupation code, and subsequently aggregated to the CEX occupational classification using ASEC weights (asecwt). For $risk^{(2)}_{i,j,t}$ we use a robot exposure measure developed by Cossu, Moro, and Rendall (2026). For $risk^{(3)}_{i,j,t}$, we use the routine task intensity index developed by Autor and Dorn (2013). Both indices are defined at the occupational level using the OCC1990dd and OCC1990 classification, which allows us to aggregate them to the broader occupational categories reported in the CEX. Aggregation is performed using employment-weights, where weights reflect the relative occupational shares within each CEX group.

To align the occupational-level risk measures with the CEX classification, we map all indicators with those derived from CPS data and those based on the routine and robotization indices to the occupational categories reported in the CEX. This is done using a consistent crosswalk between OCC1990 and OCC1990dd codes. We then group occupations into the 14 major occupational categories defined by the CEX, excluding the 15th category, which refers to members of the armed forces.[5]

2 Model

2.1 Sectoral Productivity Dynamics

Sectoral TFP decomposes into a deterministic trend $T_{i,t}$ and a stationary log-AR(1) component $\tilde{A}_{i,t}$ with mean-correction term $d_i = -\frac{1}{2}\sigma_i^2/\left(1+\rho_i\right)$, as in equations (23), (24), (25) and (26) of the main text, where $g > 0$, $\kappa > 0$, and $|\rho_i| < 1$.

ASSUMPTION 2.1: *Let, for each sector $i \in \{m, s\}$, the innovations $\nu_{i,t} \sim \mathcal{N}\left(0, \sigma_i^2\right)$ be i.i.d. over time and independent across sectors at all leads and lags, and let the*

[5] We define: Manager, Professional administrators occ1990dd>=3 & occ1990dd<=37; Teacher occ1990dd>=154 & occ1990dd<=159; Professional occ1990dd>=43 & occ1990dd<=153; occ1990dd>=160 & occ1990dd<=200; Administrative support occ1990dd>=303 & occ1990dd<=389; Sales retail occ1990dd>=274 & occ1990dd<=283; Sales, Business Goods and Services: occ1990dd>=243 & occ1990dd<=258; occ1990dd>=433 & occ1990dd<=444; occ1990dd>=445 & occ1990dd<=447; Technician Service: occ1990dd>=203 & occ1990dd<=235; Protective Service: occ1990dd>=415 & occ1990dd<=427; Private Household Service: occ1990dd>=405 & occ1990dd<=408; occ1990dd>=448 & occ1990dd<=455; Other Service: occ1990dd>=457 & occ1990dd<=458; occ1990dd>=459 & occ1990dd<=467; occ1990dd>=469 & occ1990dd<=472; occ1990dd==468; Machine or Transportation Operator, Laborer: occ1990dd>=503 & occ1990dd<=549; occ1990dd>=703 & occ1990dd<=799; occ1990dd>=803 & occ1990dd<=889; Construction Workers, Mechanics: occ1990dd>=558 & occ1990dd<=599; occ1990dd>=628 & occ1990dd<=699; Farming: occ1990dd>=473 & occ1990dd<=489; Forestry, Fishing, Groundskeeping: occ1990dd>=496 & occ1990dd<=498.

process be initialized at its invariant distribution,

$$\log \tilde{A}_{i,0} \sim \mathcal{N}\left(-\tfrac{1}{2}\tfrac{\sigma_i^2}{1-\rho_i^2}, \tfrac{\sigma_i^2}{1-\rho_i^2}\right). \tag{2.1}$$

Normality and cross-sectional independence deliver the closed-form log-normal mean and the product structure of the discretized state space, while the persistence restriction makes the mean correction the unique mean-neutral drift. The ergodic initialization is the key, since without it the convergence of the stochastic moments would shift average productivity along the transition and contaminate the deterministic trend.

PROPOSITION 2.1: *If Assumption 2.1 holds, the stochastic component of sectoral TFP is strictly stationary from the first period, with the log-productivity of every date distributed as the invariant distribution, and it is mean-neutral in levels at every date,*

$$\mathbb{E}[\tilde{A}_{i,t}] = 1 \text{ for all } t \geq 0.$$

PROOF OF PROPOSITION 2.1: Define the log-productivity of sector i as

$$\tilde{a}_{i,t} \equiv \log \tilde{A}_{i,t}. \tag{2.2}$$

By forward iteration, the log-AR(1) process can be expressed as

$$\tilde{a}_{i,t} = \rho_i^t \tilde{a}_{i,0} + \sum_{j=1}^{t} \rho_i^{t-j} d_i + \sum_{j=1}^{t} \rho_i^{t-j} \nu_{i,j},$$

which is a linear combination of independent Gaussian terms under Assumption 2.1, hence Gaussian and characterized by its first two moments. Initialized at the invariant distribution, these moments equal the invariant values in (2.1) at every date, so the process is strictly stationary from the first period. Log-normality then gives

$$\mathbb{E}\left[\tilde{A}_{i,t}\right] = \mathbb{E}\left[\exp\left(\tilde{a}_{i,t}\right)\right] = \exp\left(\mathbb{E}\left[\tilde{a}_{i,t}\right] + \frac{1}{2}\mathrm{Var}\left[\tilde{a}_{i,t}\right]\right) = 1.$$

$\square$

Together, Assumption 2.1 and Proposition 2.1 make uncertainty enter the economy as pure risk, since the stochastic component fluctuates around each deterministic trend while the mean correction offsets the Jensen term, leaving average productivity unchanged at every date.

The sectoral growth differential in the trends is transitory, decaying geometrically through the decay factor in equation (24) of the main text.

PROPOSITION 2.2: *The gross growth rates of the two sectoral trends converge to the common value*

$$\lim_{t\to\infty}\left[\frac{T_{i,t}}{T_{i,t-1}}\right] = 1+g. \tag{2.3}$$

PROOF OF PROPOSITION 2.2: The common factors in the trend ratio cancel, leaving

$$\frac{T_{i,t}}{T_{i,t-1}} = \frac{\prod_{h=1}^{t}\left(1+g+\alpha_i e^{-\kappa h}\right)}{\prod_{h=1}^{t-1}\left(1+g+\alpha_i e^{-\kappa h}\right)} = 1+g+\alpha_i e^{-\kappa t}.$$

Since the decay rate is strictly positive, the sector-specific term vanishes as $t\to\infty$, and

$$\lim_{t\to\infty}\left[\frac{T_{i,t}}{T_{i,t-1}}\right] = \lim_{t\to\infty}\left[1+g+\alpha_i e^{-\kappa t}\right] = 1+g.$$

□

Relative prices inherit the behavior of relative TFP, since the firms' FOCs tie them, as in equation (22) of the main text. Trend growth convergence, established in Proposition 2.2, is necessary but not sufficient for their ratio to converge, since equal long-run growth does not pin down the limit of the trend ratio.

PROPOSITION 2.3: *If Assumption 2.1 holds, the expected relative TFP converges to a finite constant,*

$$\lim_{t\to\infty}\mathbb{E}\left[\frac{A_{i,t}}{A_{-i,t}}\right] = \exp\left(\frac{\sigma_{-i}^2}{1-\rho_{-i}^2}\right)\cdot\lim_{t\to\infty}\left[\frac{T_{i,t}}{T_{-i,t}}\right], \tag{2.4}$$

where the limit of the trend ratio exists and is finite.

PROOF OF PROPOSITION 2.3: Since the deterministic trends are independent of the stochastic components, they factor out of the expectation at every date, so the expected relative TFP splits into a stochastic and a deterministic factor. Taking limits,

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{A_{i,t}}{A_{-i,t}}\right] = \lim_{t\to\infty} \mathbb{E}\left[\frac{\tilde{A}_{i,t}}{\tilde{A}_{-i,t}}\right] \cdot \lim_{t\to\infty}\left[\frac{T_{i,t}}{T_{-i,t}}\right].$$

By Proposition 2.1 and the cross-sectional independence of Assumption 2.1, the difference of the sectoral log-productivities is Gaussian,

$$\tilde{a}_{i,t} - \tilde{a}_{-i,t} \sim \mathcal{N}\left(\mathbb{E}\left[\tilde{a}_{i,t}\right] - \mathbb{E}\left[\tilde{a}_{-i,t}\right], \mathrm{Var}\left[\tilde{a}_{i,t}\right] + \mathrm{Var}\left[\tilde{a}_{-i,t}\right]\right),$$

so its exponential is log-normal, with constant mean

$$\mathbb{E}\left[\frac{\tilde{A}_{i,t}}{\tilde{A}_{-i,t}}\right] = \exp\left\{\mathbb{E}\left[\tilde{a}_{i,t}\right] - \mathbb{E}\left[\tilde{a}_{-i,t}\right] + \frac{1}{2}\left(\mathrm{Var}\left[\tilde{a}_{i,t}\right] + \mathrm{Var}\left[\tilde{a}_{-i,t}\right]\right)\right\} = \exp\left(\frac{\sigma_{-i}^2}{1-\rho_{-i}^2}\right).$$

This constant exceeds one because the expected reciprocal of a unit-mean log-normal variable is itself larger than one. By mean neutrality the terms of sector i cancel, while those of sector $-i$ survive, so that the constant depends on the unconditional variance of the second.

For the deterministic part, combining the two product operators into a single one and manipulating, we obtain

$$\frac{T_{i,t}}{T_{-i,t}} = \frac{\prod_{h=1}^{t}\left(1+g+\alpha_i e^{-\kappa h}\right)}{\prod_{h=1}^{t}\left(1+g+\alpha_{-i} e^{-\kappa h}\right)} = \prod_{h=1}^{t}\left(\frac{1+g+\alpha_i e^{-\kappa h}}{1+g+\alpha_{-i} e^{-\kappa h}}\right) = \prod_{h=1}^{t}\left[1+\frac{\left(\alpha_i-\alpha_{-i}\right)e^{-\kappa h}}{1+g+\alpha_{-i} e^{-\kappa h}}\right].$$

The logarithm of this product is the sum

$$\log\left\{\prod_{h=1}^{t}\left[1+\frac{\left(\alpha_i-\alpha_{-i}\right)e^{-\kappa h}}{1+g+\alpha_{-i} e^{-\kappa h}}\right]\right\} = \sum_{h=1}^{t}\log\left[1+\frac{\left(\alpha_i-\alpha_{-i}\right)e^{-\kappa h}}{1+g+\alpha_{-i} e^{-\kappa h}}\right],$$

whose terms are bounded by a geometric series under a strictly positive decay rate κ, so the sum converges and the limit of the trend ratio exists and is finite.

$\square$

REMARK 2.1: As $t \to \infty$, a first-order expansion of the sum in the proof yields

$$\sum_{h=1}^{\infty}\log\left[1+\frac{\left(\alpha_i-\alpha_{-i}\right)e^{-\kappa h}}{1+g+\alpha_{-i} e^{-\kappa h}}\right] \approx \sum_{h=1}^{\infty}\frac{\left(\alpha_i-\alpha_{-i}\right)e^{-\kappa h}}{1+g+\alpha_{-i} e^{-\kappa h}}.$$

Neglecting the small sector-specific term in the denominator, the sum reduces to a geometric series,

$$\sum_{h=1}^{\infty} \frac{(\alpha_i - \alpha_{-i})}{1+g} e^{-\kappa h} = \frac{\alpha_i - \alpha_{-i}}{(1+g)\left(e^{\kappa} - 1\right)}.$$

Consequently, the limiting trend ratio is approximated in closed form as

$$\lim_{t\to\infty} \left[\frac{T_{i,t}}{T_{-i,t}}\right] = \prod_{h=1}^{\infty} \left[1 + \frac{(\alpha_i - \alpha_{-i})\, e^{-\kappa h}}{1+g+\alpha_{-i} e^{-\kappa h}}\right] \approx \exp\left(\frac{\alpha_i - \alpha_{-i}}{(1+g)\left(e^{\kappa} - 1\right)}\right),$$

and the expected long-run relative TFP is approximately

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{A_{i,t}}{A_{-i,t}}\right] \approx \exp\left(\frac{\sigma_{-i}^2}{1-\rho_{-i}^2} + \frac{\alpha_i - \alpha_{-i}}{(1+g)\left(e^{\kappa} - 1\right)}\right).$$

2.2 Inter-temporal Allocation - Bellman principle

At the intertemporal stage the household chooses composite consumption and next-period capital to maximize expected lifetime utility, trading current consumption against the return to capital under stochastic sectoral productivity. Substituting the nominal investment flow from (18) of the main text into the household budget constraint, and combining it with the aggregate law of motion for capital (16) of the main text, the household's capital accumulates, in units of the composite investment good, as

$$k_{t+1} = k_t \left(\frac{r_t}{p_{\mathcal{I},t}} + 1 - \delta\right) + \frac{w_t}{p_{\mathcal{I},t}} n_t - \frac{p_{\mathcal{C},t}}{p_{\mathcal{I},t}} \mathcal{C}_t - \frac{p_{m,t}}{p_{\mathcal{I},t}} m + \frac{p_{s,t}}{p_{\mathcal{I},t}} s. \tag{2.5}$$

Let $z_t = (A_{m,t}, A_{s,t})$ be the productivity state. Because the transition is non-autonomous, the value function is time-dependent, and the household solves the Bellman equation

$$V_t\left(k_t, z_t\right) = \max_{k_{t+1}\geq 0,\, \mathcal{C}_t \geq 0} \left\{u(\mathcal{C}_t) + \beta \mathbb{E}\left[V_{t+1}\left(k_{t+1}, z_{t+1}\right) \mid z_t\right]\right\},$$

subject to the accumulation constraint (2.5), taking as given the price indices for investment (17) and consumption (14) and the equilibrium relative price system (22) of the main text. The first-order and envelope conditions of this problem yield the

Euler equation, which characterizes the household's optimal savings and consumption decision,

$$u'\left(\mathcal{C}_t\right)\frac{p_{\mathcal{I},t}}{p_{\mathcal{C},t}}=\beta\mathbb{E}\left[u'\left(\mathcal{C}_{t+1}\right)\frac{p_{\mathcal{I},t+1}}{p_{\mathcal{C},t+1}}\left(\frac{r_{t+1}}{p_{\mathcal{I},t+1}}+1-\delta\right)\mid z_t\right]. \tag{2.6}$$

Together with the transversality condition on the discounted marginal value of capital, the Euler equation fully characterizes the optimum, since the objective is strictly concave and the constraint set is convex.

3 Solution Strategy

Non-homothetic preferences and asymmetric trend growth make the problem non-autonomous along the transition, so value and policy functions carry an explicit time dependence and no time-invariant steady state anchors the solution. The asymptotic phase, in which the problem becomes autonomous, is therefore solved first, and its policies serve as the terminal condition of a backward solution along the time dimension. Expectations over future productivity states are computed with the Markov-chain approximation of Rouwenhorst (1995), discussed also by Kopecky and Suen (2010).

Following Gadea, Gómez-Loscos, and Pérez-Quirós (2020), who identify structural discontinuities in U.S. output volatility, the variance of innovations changes at a limited number of dates, each change defining a volatility regime that the household believes to be permanent. The shift is therefore unanticipated, and consistency requires that every step of the backward pass use the transition law of the regime in place at the date of the choice, which amounts to solving each regime as a self-contained problem over the whole horizon. The regimes therefore meet only in the forward simulation, where at a switch date the productivity state realized under the old law is mapped onto the state space of the new one by a bridge matrix.

3.1 The Asymptotic Balanced Growth Path

By Propositions 2.2 and 2.3, relative sectoral TFP is asymptotically stationary, being the product of a stationary stochastic ratio and a trend ratio with a finite limit.

ASSUMPTION 3.1: *Let* $\tilde{z}_t=\left(\tilde{A}_{m,t},\tilde{A}_{s,t}\right)$ *be the exogenous state, and let it take values in a finite set* $\mathcal{Z}\subset\mathbb{R}^2_{++}$ *and follow an irreducible and aperiodic Markov chain,* *whose invariant distribution and persistence are those of Assumption 2.1. Let the feasible set impose a consumption floor $\varepsilon_t > 0$ on consumption so that $\mathcal{C}_t \geq \varepsilon_t$ at every date.*

Both restrictions make the limit problem tractable with classical recursive methods. A finite exogenous state turns conditional expectations into finite sums, while the lower bound on consumption bounds period utility from below, and delivers the lower end of the capital domain, where by the Inada conditions it never binds at the optimum.

PROPOSITION 3.1: *Suppose Assumptions 2.1 and 3.1 hold. Then there exists a unique scale factor X_t such that, in the limit, every variable of the economy divided by it is a time-invariant function of normalized capital $\hat{k}_t$ and the exogenous state $\tilde{z}_t$. The limit problem so obtained admits a unique value function and unique time-invariant policies, characterized by the normalized stationary Euler equation. That pair admits an invariant distribution, at which the normalized economy is strictly stationary and level variables grow at the gross rate of the scale factor, so that the existence of an asymptotic stochastic balanced growth path (BGP) is guaranteed.*

PROOF OF PROPOSITION 3.1: Denote by a hat every level variable divided by the scale factor, so that for a generic level variable $\mathcal{B}_t$,

$$\hat{\mathcal{B}}_t \equiv \frac{\mathcal{B}_t}{X_t},$$

we can write capital stock as $k_t = \hat{k}_t X_t$. Perfect factor mobility equalizes the value of the marginal product across sectors, so aggregate output can be written in manufacturing units, and substituting the trend-stochastic decomposition of sectoral TFP, real normalized output reads

$$\frac{\hat{\mathcal{Y}}_t^N}{p_{\mathcal{I},t}} \equiv \frac{\mathcal{Y}_t^N}{p_{\mathcal{I},t} X_t} = \frac{p_{m,t}}{p_{\mathcal{I},t}} \tilde{A}_{m,t} \hat{k}_t^{\theta} N_t^{1-\theta} \cdot T_{m,t} X_t^{\theta-1},$$

which depends explicitly on the trend only through the last factor $T_{m,t} X_t^{\theta-1}$ and is therefore trend-invariant for the unique value

$$X_t = (T_{m,t})^{\frac{1}{1-\theta}}, \ \forall t \geq 0, \tag{3.1}$$

whose gross growth rate $X_{t+1}/X_t \equiv (1+\mu_{X,t})$ converges to $(1+g)^{\frac{1}{1-\theta}}$ as t grows large, by Proposition 2.2.

Relative prices inherit the asymptotic stationarity of relative TFP by equation (22) of the main text, and since both price indices are homogeneous of degree one in sectoral prices, every ratio among prices and indices is homogeneous of degree zero and depends on the relative price alone. Writing $\tau_t \equiv p_{s,t}/p_{m,t}$,

$$\frac{p_{m,t}}{p_{\mathcal{I},t}} = \left[\varphi_m + \varphi_s \tau_t^{1-\eta}\right]^{-\frac{1}{1-\eta}}, \quad \frac{p_{s,t}}{p_{\mathcal{I},t}} = \left[\varphi_s + \varphi_m \tau_t^{\eta-1}\right]^{-\frac{1}{1-\eta}}, \quad \frac{p_{\mathcal{C},t}}{p_{\mathcal{I},t}} = \frac{\left(\omega_m^{\omega_m}\omega_s^{\omega_s}\right)^{-1}\tau_t^{\omega_s}}{\left[\varphi_m + \varphi_s \tau_t^{1-\eta}\right]^{\frac{1}{1-\eta}}}.$$

All price ratios entering the household problem are therefore functions of stationary processes alone.

Analogously, substituting the normalized capital stock and the decomposition of sectoral TFP into the firms' first-order conditions, and using (3.1), these become

$$\frac{r_t}{p_{\mathcal{I},t}} = \frac{p_{m,t}}{p_{\mathcal{I},t}}\theta \tilde{A}_{m,t}\hat{k}_t^{\theta-1}, \qquad \frac{\hat{w}_t}{p_{\mathcal{I},t}} = \frac{p_{m,t}}{p_{\mathcal{I},t}}(1-\theta)\tilde{A}_{m,t}\hat{k}_t^{\theta},$$

time-invariant functions of stationary processes and of normalized capital.

Dividing the law of motion (2.5) by the scale factor and defining normalized consumption and subsistence terms as $\hat{\mathcal{C}}_t \equiv \mathcal{C}_t/X_t$, $\hat{m}_t \equiv m/X_t$, $\hat{s}_t \equiv s/X_t$, the accumulation constraint in normalized units reads

$$\hat{k}_{t+1}(1+\mu_{X,t}) = \hat{k}_t\left(\frac{r_t}{p_{\mathcal{I},t}} + 1 - \delta\right) + \frac{\hat{w}_t}{p_{\mathcal{I},t}}n_t - \frac{p_{\mathcal{C},t}}{p_{\mathcal{I},t}}\hat{\mathcal{C}}_t - \frac{p_{m,t}}{p_{\mathcal{I},t}}\hat{m}_t + \frac{p_{s,t}}{p_{\mathcal{I},t}}\hat{s}_t. \tag{3.2}$$

The subsistence terms are constants, so their weight declines as the economy grows and their normalized counterparts vanish as the scale factor diverges. They drop out of both (3.2) and the normalized counterpart $\hat{\mathcal{C}}_t$ of eq. (11) of the main text, which in the limit becomes homothetic in the two sectoral goods. This, together with the convergence of the growth factor $1+\mu_{X,t}$ to its limit $(1+g)^{\frac{1}{1-\theta}}$, makes (3.2) a function of stationary processes, normalized capital and normalized consumption. What in Kongsamut, Rebelo, and Xie (2001) obstructs balanced growth and has to be removed by a knife-edge restriction on preference parameters here dies out on its own, so that the asymptotic phase is reached through two distinct forces, the vanishing of non-homotheticity and the convergence of the sectoral technologies.

Period utility is homogeneous of degree $1-\gamma$ in consumption composite, so that

$u(\mathcal{C}_t) = X_t^{1-\gamma} u(\hat{\mathcal{C}}_t)$ and the value function inherits the same scaling. Defining

$$v_t\left(\hat{k}_t, \tilde{z}_t\right) \equiv \frac{V_t\left(k_t, z_t\right)}{X_t^{1-\gamma}},$$

where the state collects the stochastic components of productivity alone, the deterministic trends having been absorbed by the normalization, the Bellman equation of Section 2.2 becomes

$$v_t\left(\hat{k}_t, \tilde{z}_t\right) = \max_{\hat{k}_{t+1} \geq 0, \hat{\mathcal{C}}_t \geq \frac{\varepsilon_t}{X_t}} \left\{ u(\hat{\mathcal{C}}_t) + \left(1 + \mu_{X,t}\right)^{1-\gamma} \beta \cdot \mathbb{E}\left[v_{t+1}\left(\hat{k}_{t+1}, \tilde{z}_{t+1}\right) \mid \tilde{z}_t \right] \right\} \tag{3.3}$$

subject to (3.2), where the consumption floor is proportional to the scale factor, $\varepsilon_t \equiv \varsigma X_t$ so that the normalized problem carries the constant lower bound $\hat{\mathcal{C}}_t \geq \varsigma > 0$ and consequently period utility is bounded at every date t. This implies the Euler equation

$$u'\left(\hat{\mathcal{C}}_t\right) \frac{p_{\mathcal{I},t}}{p_{\mathcal{C},t}} = \beta \left(1 + \mu_{X,t}\right)^{-\gamma} \cdot \mathbb{E}\left[u'\left(\hat{\mathcal{C}}_{t+1}\right) \frac{p_{\mathcal{I},t+1}}{p_{\mathcal{C},t+1}} \left(\frac{r_{t+1}}{p_{\mathcal{I},t+1}} + 1 - \delta \right) \mid \tilde{z}_t \right], \tag{3.4}$$

where the exponent on the growth factor $-\gamma$ differs from the one in the Bellman equation because the rescaling passes through marginal utility.

The scale factor passes through the consumption composite without residuals, since ($\omega_m + \omega_s = 1$), and the sectoral compositions of consumption and investment are functions of prices and aggregates alone, so the limit problem is a canonical one-sector stochastic growth model with trend growth, and standard theorems apply.

Since normalized output is homogeneous of degree $\theta < 1$ in capital while the consumption floor of Assumption 3.1 is constant in normalized units, below a strictly positive level $\hat{k}_{\min}$ no feasible plan delivers $\hat{\mathcal{C}}_t \geq \varsigma$ and the feasible set of (3.2) is empty. Above a maximum sustainable level $\hat{k}_{\max}$ the cost of carrying the stock, which grows at rate $\delta + \mu_{X,t}$, exceeds what output delivers, and capital declines. Normalized capital therefore lives in a compact interval $\left[\hat{k}_{\min}, \hat{k}_{\max}\right] \subset \mathbb{R}_{++}$ on which period utility, evaluated at feasible choices, is bounded, continuous, continuously differentiable, strictly concave and strictly increasing in both states, and the feasible correspondence is non-empty, compact-valued, continuous, convex and increasing in capital.

For the problem to admit a solution the discount factor of the normalized Bellman equation must lie in the unit interval, which holds at every date since

$$\hat{\beta}_t \equiv \beta \left(1 + \mu_{X,t}\right)^{1-\gamma} < \beta < 1,$$

because the household is risk averse and impatient and trend growth is positive, and $\hat{\beta}_t$ converges to $\beta \left(1 + g\right)^{\frac{1-\gamma}{1-\theta}}$ as $t \to \infty$.

Assumptions 9.4 to 9.12 of Stokey, Lucas, and Prescott (1989) are therefore satisfied. Their Theorem 9.6 delivers a unique value function, to which the iteration converges geometrically at the asymptotic rate $\hat{\beta}$ from any continuous initial guess, which underlies the numerical solution of Section 4.

The optimal policies are time-invariant functions of the pair $\left(\hat{k}_t, \tilde{z}_t\right)$, and initialized at the invariant distribution the pair is a strictly stationary process. Level variables are the product of their normalized counterparts and the scale factor, and therefore grow at its gross rate, which converges by Proposition 2.2. The economy is thus on an asymptotic stochastic BGP.

□

3.2 Solving the Stochastic Transitional Dynamics

For $t < T$, the growth rate of the trend ratio has not yet converged, so the policy functions depend on time. The recursive structure is otherwise that of the limit problem, with the same states, controls, objective and accumulation constraint.

The problem is solved by backward induction from $t = T$, where the terminal condition is the limit policy of Proposition 3.1, backward to $t = 0$. At each date expectations over future productivity are taken under the transition law of the volatility regime that the household expects to last forever. This delivers, for each volatility regime, a sequence of date-dependent policies

$$\left\{\hat{\mathcal{C}}_t^* \left(\hat{k}, \tilde{z}\right), \hat{k}_{t+1}^* \left(\hat{k}, \tilde{z}\right)\right\}_{t=0}^{T},$$

which satisfy the Euler equation (3.4) subject to (3.2) at every date and describe the optimal stochastic transition toward the balanced growth path. The aggregate normalized decisions are then mapped back into their sectoral components, through the static optimality conditions, which makes the structural transformation observable along the transition.

3.3 Unanticipated Volatility Shifts and Second-Moment Persistence

Let $r = 1, \ldots, R$ index the volatility regimes, each with innovation variances $\boldsymbol{\sigma}^{(r)} = \left\{\sigma_m^{(r)}, \sigma_s^{(r)}\right\}$, correction term $d_i^{(r)}$ and log-productivity process $\tilde{a}_{i,t}^{(r)}$, initialized at the invariant distribution they imply in accordance with Assumption 2.1. Each regime is solved as in the previous subsections, yielding R independent policy sequences over $t = 0, \ldots, T$, each consistent with the Bellman principle conditional on the household's belief that the volatility regime is permanent.

The economy crosses from one regime to the next one only in the forward simulation, when a trajectory reaches a switch date. At a switch between $t^* - 1$ and t^*, the capital k_{t^*} was chosen under the old regime and is predetermined, so the surprise operates exclusively through the productivity state, expected under the law associated with $\boldsymbol{\sigma}^{(r)}$ and realized under the law associated with $\boldsymbol{\sigma}^{(r+1)}$. The inherited position transmits information through its first moment and through its second.

Define the regime-centered log-productivity as

$$\xi_{i,t}^{(r)} \equiv \tilde{a}_{i,t}^{(r)} - \mu_i^{(r)}, \qquad \mu_i^{(r)} \equiv \mathbb{E}^{(r)}\left[\tilde{a}_{i,t}^{(r)}\right] = -\frac{1}{2}\frac{\left(\sigma_i^{(r)}\right)^2}{1-\rho_i^2},$$

where $\mathbb{E}^{(r)}$ denotes the expectation under the invariant distribution of regime r, so that within the regime $\xi_{i,t}^{(r)}$ follows a zero-mean auto-regressive process and levels are recovered as

$$\tilde{A}_{i,t}^{(r)} = \exp\left(\xi_{i,t}^{(r)} + \mu_i^{(r)}\right). \tag{3.5}$$

ASSUMPTION 3.2: *Let the switch from regime r to regime $r+1$ occur between $t^* - 1$ and t^*. The transition of the productivity state across the switch is governed by the auto-regressive process with the new innovation variance, applied to the inherited centered state,*

$$\xi_{i,t^*}^{(r+1)} = \rho_i \xi_{i,t^*-1}^{(r)} + \nu_{i,t^*}^{(r+1)}, \qquad \nu_{i,t^*}^{(r+1)} \sim \mathcal{N}\left(0, \left(\sigma_i^{(r+1)}\right)^2\right),$$

so that the level of the stochastic component is recovered under the new regime's mean.

The auto-regressive process of Section 2.1 does not by itself determine how the state crosses a switch, so Assumption 3.2 is a modeling choice, and among the possible treatments of the inherited state we consider the three natural ones. The state can be carried over in levels, with $\tilde{a}_{i,t^*}^{(r+1)} = \rho_i \tilde{a}_{i,t^*-1}^{(r)} + d_i^{(r+1)} + \nu_{i,t^*}^{(r+1)}$, in which case both moments are dragged into the new regime, the first decaying at rate ρ_i and the second at rate ρ_i^2 . Alternatively, the state can be redrawn at $t^*$from the invariant distribution of the new regime, so that $\xi_{i,t^*}^{(r+1)} \sim \mathcal{N}\left(0, \left(\sigma_i^{(r+1)}\right)^2 / (1-\rho_i^2)\right)$ independently of the state realized under the old one. Mean neutrality in levels is then exact at every date, but the information accumulated under the old volatility is entirely lost.

Alternatively, as we assume, the household carries over its deviation from the mean of the regime it has been living in, so that the surprise falls on the state without canceling that history. We adopt the last because it is the only one under which log productivity stays mean-neutral at every date while the dispersion inherited from the previous regime survives the shift. The ergodic initialization of Assumption 2.1 applies within each regime and not to the switch, since the state that crosses it comes from a different invariant distribution. The moments at the switch date must therefore be characterized directly, which the following proposition does.

PROPOSITION 3.2: *Under Assumptions 2.1 and 3.2, conditional on the realized state $\xi_{i,t^*-1}^{(r)}$, the state at the switch date is distributed as*

$$\xi_{i,t^*}^{(r+1)} \mid \xi_{i,t^*-1}^{(r)} \sim \mathcal{N}\left(\rho_i \xi_{i,t^*-1}^{(r)}, \left(\sigma_i^{(r+1)}\right)^2\right).$$

Unconditionally the centered state has zero mean at every date, while its variance at the switch date is

$$Var\left[\xi_{i,t^*}^{(r+1)}\right] = \rho_i^2 \frac{\left(\sigma_i^{(r)}\right)^2}{1-\rho_i^2} + \left(\sigma_i^{(r+1)}\right)^2,$$

which differs from the invariant variance of the new regime whenever the two innovation variances differ. From t^*+1 onwards the recursion of the new regime applies, so the distance from its invariant variance is multiplied by ρ_i^2 at every date and vanishes geometrically.

PROOF OF PROPOSITION 3.2: Conditional on the realized state, the inherited term in Assumption 3.2 is deterministic, so the conditional moments under the law of the assumption are

$$\mathbb{E}\left[\xi_{i,t^*}^{(r+1)} \mid \xi_{i,t^*-1}^{(r)}\right] = \rho_i \xi_{i,t^*-1}^{(r)},$$
$$\operatorname{Var}\left[\xi_{i,t^*}^{(r+1)} \mid \xi_{i,t^*-1}^{(r)}\right] = \operatorname{Var}\left[\nu_{i,t^*}^{(r+1)}\right] = \left(\sigma_i^{(r+1)}\right)^2,$$

and normality follows from that of the innovation. By Assumption 2.1 the inherited state is drawn from the invariant distribution of regime r, so its unconditional moments become

$$\mathbb{E}\left[\xi_{i,t^*}^{(r+1)}\right] = \rho_i \mathbb{E}\left[\xi_{i,t^*-1}^{(r)}\right] + \mathbb{E}\left[\nu_{i,t^*}^{(r+1)}\right] = 0,$$
$$\operatorname{Var}\left[\xi_{i,t^*}^{(r+1)}\right] = \rho_i^2 \operatorname{Var}\left[\xi_{i,t^*-1}^{(r)}\right] + \operatorname{Var}\left[\nu_{i,t^*}^{(r+1)}\right] = \rho_i^2 \frac{\left(\sigma_i^{(r)}\right)^2}{1-\rho_i^2} + \left(\sigma_i^{(r+1)}\right)^2,$$

which coincides with the invariant variance of regime $r+1$, if and only if the two innovation variances are equal.

After the switch the variance follows the recursion of regime $r+1$, so that s periods later

$$\operatorname{Var}\left[\xi_{i,t^*+s}^{(r+1)}\right] = \rho_i^{2(s+1)} \frac{\left(\sigma_i^{(r)}\right)^2}{1-\rho_i^2} + \left(\sigma_i^{(r+1)}\right)^2 \sum_{j=0}^{s} \rho_i^{2j} \xrightarrow[s\to\infty]{} \frac{\left(\sigma_i^{(r+1)}\right)^2}{1-\rho_i^2},$$

that is, the variance converges geometrically at rate ρ_i^2 to the ergodic variance of the new regime.

□

The variance inherited at the switch date therefore differs from the invariant one, and the correction term of the new regime, which is calibrated on the latter, does not offset it. This leaves a trace in the level of the stochastic component.

COROLLARY 3.1: *Under Assumptions 2.1 and 3.2, the stochastic component of productivity is mean-neutral in logs at every date. In levels, since the correction term of regime $r+1$ offsets exactly its invariant variance, the log-normal mean reduces to one half of the distance between the actual and the invariant variance, so that for*

$s \geq 0$, this is

$$\mathbb{E}\left[\tilde{A}_{i,t^*+s}^{(r+1)}\right] = \exp\left(\mu_i^{(r+1)} + \frac{1}{2}\,Var\left[\xi_{i,t^*+s}^{(r+1)}\right]\right),$$
$$= \exp\left(\frac{\rho_i^{2(s+1)}}{2}\left[\frac{\left(\sigma_i^{(r)}\right)^2}{1-\rho_i^2} - \frac{\left(\sigma_i^{(r+1)}\right)^2}{1-\rho_i^2}\right]\right),$$

which differs from unity whenever the two innovation variances differ and converges to unity at rate ρ_i^2.

Mean neutrality in levels is therefore an ergodic property, exact within each regime and restored only gradually after a switch. The deviation is quantitatively small and vanishes within a few periods, and it is the price of a convention that keeps the centered process mean-zero at every date, a useful property in numerical solutions which discretize the processes on finite grids, while at the same time preserving the memory of the regime the economy leaves instead of resetting it to a new universe at the switch date.

4 The Numerical Algorithm

The model admits no closed-form solution, so the numerical algorithm solves the household's problem and simulates the economy in six stages.

4.1 Discretization of the Exogenous State

As set out in the solution strategy, the exogenous state is discretized sector by sector with the Rouwenhorst (1995) method, applied to the centered log-productivity process $\xi_{i,t} \equiv \tilde{a}_{i,t} - \mu_i$ on the symmetric grid with the same number of nodes in the two sectors ($N_m = N_s = N$),

$$\mathcal{G}_i^{\xi} = \left\{\xi_i^{(1)}, \xi_i^{(2)}, ..., \xi_i^{(N)}\right\},$$

where $\xi_i^{(j)} = -\psi_i + (j-1)\,\Delta_i$, with $\psi_i = \sigma_{\xi_i}\sqrt{N-1}$ and $\Delta_i = \frac{2\psi_i}{N-1}$, and an associated transition matrix $\mathbf{\Pi}_i \in \mathbb{R}^{N\times N}$. Its support is proportional to the stationary standard deviation of the continuous process, $\sigma_{\xi_i} = \sigma_i/\sqrt{1-\rho_i^2}$, ensuring that the unconditional variance of the Markov chain closely matches that of the AR(1). The transition matrix $\mathbf{\Pi}_i$ is built by a recursion that starts from a base 2×2 symmetric matrix and preserves symmetry, persistence and the second moments of the AR(1) for any N. It is parameterized by $q_i = (\rho_i + 1)/2$, so that the implied first-order auto-correlation of the discrete process matches ρ_i exactly.

Reintroducing the ergodic mean delivers the sectoral grids in levels $\mathcal{G}_i^{\tilde{A}}$, whose Cartesian product $\mathcal{G}_m^{\tilde{A}} \times \mathcal{G}_s^{\tilde{A}}$ yields the joint grid $\mathcal{G}^{\tilde{z}} = \left\{\tilde{z}^{(1)}, \tilde{z}^{(2)}, ..., \tilde{z}^{(N^2)}\right\}$ which discretizes the exogenous state space $\tilde{\mathcal{Z}}$. Since sectoral innovations are independent, the joint transition matrix is the Kronecker product

$$\mathbf{\Pi}_{\tilde{z}} = \mathbf{\Pi}_m \otimes \mathbf{\Pi}_s, \quad \mathbf{\Pi}_{\tilde{z}} \in \mathbb{R}^{N^2 \times N^2}.$$

Conditional expectations are then matrix–vector products rather than numerical integrations. The grid inherits the mean neutrality of Proposition 2.1 up to the error of matching two moments only, and within a regime, the grid and transition matrix are fixed, so the time dependence of the transition phase is confined to the deterministic trends.

4.2 Discretization of the Endogenous state

The proof of Proposition 3.1 confines normalized capital to a compact interval, since the feasible set is empty below a strictly positive level and capital declines above a maximum sustainable one. We discretize that interval on the one-dimensional grid

$$\mathcal{G}^{\hat{k}} = \left\{\hat{k}^{(1)}, \hat{k}^{(2)}, ..., \hat{k}^{(N_{\hat{k}})}\right\}.$$

The upper bound $\hat{k}_{\max}$ is the deterministic asymptotic steady state augmented by a buffer that accommodates stochastic realizations and precautionary saving. The lower bound is the counterpart of the consumption floor of Assumption 3.1, and we set $\hat{k}_{\min}$ to the smallest normalized capital for which the budget constraint admits stictly positive consumption at every exogenous state, which keeps the Bellman problem well defined on the whole grid and rules out corner solutions driven by infeasible subsistence.

The grid is spaced non-uniformly, with a power rule that concentrates nodes at low $\hat{k}$, where the policy functions are most curved.

4.3 Discretization of the Non-Autonomous Block and the Euler Equation

During the transition each variance regime is a self-contained universe in which the stochastic block is stationary, as established in Section 2.1, and preserved by the discretization of Section 4.1, so its grid $\mathcal{G}^{\tilde{z}}$ and its transition matrix are fixed throughout, and the only source of time dependence in the level grid are the deterministic trends $\mathcal{T}_t = (T_{m,t}, T_{s,t})$, which are common to all regimes. For each date the level productivity grid is obtained node by node as

$$A_t^{(j)} = \tilde{z}^{(j)} \odot \mathcal{T}_t,$$

where $\odot$ is the Hadamard product, which yields the sequence of time-indexed grids

$$\left\{\mathcal{G}_t^A\right\}_{t=0}^T = \left\{A_t^{(1)}, A_t^{(2)}, ..., A_t^{(N^2)}\right\}_{t=0}^T.$$

From these, together with the growth factor $1 + \mu_{X,t}$ and the normalized subsistence terms $\hat{m}_t$ and $\hat{s}_t$, all equilibrium prices and factor returns entering the Euler equation and the normalized budget constraint defined in Section 3.1 are determined at each date and state.

The optimal policy is therefore defined on a three-dimensional space discretized by the three grids, $\mathcal{G}^{\hat{k}} \times \mathcal{G}^{\tilde{z}} \times \mathcal{G}^t$. Non-homotheticity and asymmetric trend growth are what make the third dimension necessary, since the optimal policy is not the same function at every date, and the same discretized space accommodates both phases of the solution. The Euler equation (3.4) is therefore discretized on all three dimensions. Indexing a node by $\left(\hat{k}^{(m)}, \tilde{z}^{(n)}, t\right)$, with next-period capital $\hat{k}'$ implied by the normalized budget constraint (3.2) and the conditional expectation replaced by the sum over the columns n' of row n of the transition matrix (the row indexing the current exogenous state and the columns the possible future ones), the optimal policy satisfies

$$
\begin{aligned}
u'\left[\hat{\mathcal{C}}^{*}\left(\hat{k}^{(m)}, \tilde{z}^{(n)}, t\right)\right] \cdot \frac{p_{\mathcal{I}}}{p_{\mathcal{C}}}\left(\tilde{z}^{(n)}, t\right) = & \beta\left[1+\mu_X(t)\right]^{-\gamma} \cdot \sum_{n'=1}^{N^2}\left[\mathbf{\Pi}_{\tilde{z}}\right]_{n,n'} \\
& \cdot\left\{u'\left[\hat{\mathcal{C}}^{*}\left(\hat{k}', \tilde{z}^{(n')}, t+1\right)\right] \cdot \frac{p_{\mathcal{I}}}{p_{\mathcal{C}}}\left(\tilde{z}^{(n')}, t+1\right)\right. \\
& \left.\cdot\left[\frac{r}{p_{\mathcal{I}}}\left(\hat{k}', \tilde{z}^{(n')}, t+1\right)+1-\delta\right]\right\}.
\end{aligned} \tag{4.1}
$$

4.4 The Policy Function Iteration (for $t \geq T$)

After having discretized the environment, we compute the stationary policy. The horizon T at which the terminal condition is imposed is set endogenously as the first date at which the normalized subsistence terms become negligible, $\hat{m}_t + \hat{s}_t < 10^{-4}$, so that from T onwards non-homotheticity has died out and the subsistence terms drop out of the normalized budget constraint (3.2), which implies that the problem is autonomous up to the residual deterministic trends. Those residuals are not exactly zero at T, since the sectoral growth differential decays at the slow rate κ, but the error they leave in the terminal condition is damped by the effective discount factor $\hat{\beta}_t < \beta < 1$ at every step of the backward induction, so that what survives at a generic date t is scaled by $\prod_{j=t}^{T-1} \hat{\beta}_j$ and is negligible over the horizon of the simulation. From T onwards the policy can be treated as independent of the date, so that $\hat{\boldsymbol{\mathcal{C}}}^{*}\left(\hat{k}, \tilde{z}, T\right) = \hat{\boldsymbol{\mathcal{C}}}^{*}\left(\hat{k}, \tilde{z}, T+1\right)$ up to the tolerance above, and the discretized Euler equation (4.1) becomes a fixed point in the policy alone. We solve it by iterating from a candidate policy until convergence. At each node the equation is solved by root-finding. For a candidate consumption on the left-hand side the implied next-period capital $\hat{k}'$ is identical across all future states, over which the right-hand side is averaged using row n of $\mathbf{\Pi}_{\tilde{z}}$. Since $\hat{k}'$ does not generally fall on a capital node, the future policy $\hat{\boldsymbol{\mathcal{C}}}^{*}\left(\hat{k}', \tilde{z}^{(n')}, T+1\right)$ and the return $\frac{\boldsymbol{r}}{\boldsymbol{p_{\mathcal{I}}}}\left(\hat{k}', \tilde{z}^{(n')}, T+1\right)$ are evaluated by interpolation (linear or cubic splines).

The stationary policy is computed regime by regime, each with its own innovation variances, since each regime r is solved as a self-contained problem over the whole horizon.

4.5 The Backward Induction (for $t < T$)

Having characterized the optimal stationary policy $\hat{\boldsymbol{\mathcal{C}}}^{*}\left(\hat{k}, \tilde{z}, T\right)$ in the asymptotic phase, we take it as the terminal condition and recover by backward induction the optimal policy at every earlier date, down to the first period, within each regime. At each date the policy solves the same discretized Euler equation (4.1), but here it is no longer a fixed point, since the policy at $t+1$ is known from the previous step, so that each date requires a single solution rather than an iteration to convergence.

What changes with respect to the autonomous phase is that the deterministic trends are still active, so prices, the real return and the growth factor are specific to the date under consideration. Root-finding and interpolation proceed as above. Propagating to $t = 0$ delivers, for each volatility regime r, the sequence of optimal policies

$$\left\{\hat{\boldsymbol{\mathcal{C}}}_t^{*,(r)}, \hat{\boldsymbol{k}'}_t^{*,(r)}\right\}_{t=0}^{T}.$$

4.6 Forward Simulation

The final stage simulates the economy forward using the optimal policy sequences $\left\{\hat{\boldsymbol{\mathcal{C}}}_t^{*}, \hat{\boldsymbol{k}'}_t^{*}\right\}_{t=0}^{T}$. The initial capital K_0 is set deterministically, representing the economy's initial development stage, and normalized as $\hat{k}_0 = K_0/X_0$. The initial productivity state cannot be assigned arbitrarily. To keep $\mathbb{E}[\tilde{A}_{i,t}] = 1$ from $t = 0$, up to the discretization error of Section 4.1, the state $\tilde{z}_0$ is drawn independently for each path from the ergodic distribution $\boldsymbol{\Pi}_{\tilde{z}}^{\infty}$ of the chain, so that the initial cross-sectional dispersion offsets the Jensen term. The simulation then proceeds iteratively for $t = 0, \ldots, T$.

Given the current state, the stored policies are interpolated at $(\hat{k}_t, \tilde{z}_t)$, which yields normalized consumption, next-period capital and the sectoral allocations embedded in the intratemporal solution.

Levels are recovered by multiplying the normalized quantities by the scale factor X_t applied after interpolation to reduce computation. Capital then moves to $\hat{k}_{t+1}^{*} = \hat{k'}^{*}\left(\hat{k}_t, \tilde{z}_t, t\right)$, and the next state $\tilde{z}_{t+1}$ is drawn from the row of $\boldsymbol{\Pi}_{\tilde{z}}$ corresponding to $\tilde{z}_t$.

When $t+1$ is a switch date the volatility regime changes. Consistently with Section 3.3, the choice at t is still made with the policies of the old regime r, since the shift is unanticipated, but the state $\tilde{z}_{t+1}$ must be drawn from the distribution implied by the new one, and from that date onwards the simulation proceeds with the policies of the new regime $r+1$. This requires a matrix mapping the states of the old regime onto the possible states of the new one, which we call the bridge $\mathbf{\Pi}_{\tilde{z}}^{(r+1|r)}$. By Proposition 3.2 the centered state at the switch is conditionally normal, with mean $\rho_i \xi_{i,t^*-1}^{(r)}$ and variance $\left(\sigma_i^{(r+1)}\right)^2$, and that continuous distribution has to be projected onto the nodes of the new grid. The two centered grids share a zero mean but differ in amplitude, since the half-width $\psi_i^{(r)} = \sigma_{\xi_i^{(r)}}\sqrt{N-1}$ depends on the volatility of the regime, so their nodes do not coincide and Rouwenhorst's method, which assumes that the source and target grids coincide, does not apply. We therefore project the conditional density with the Tauchen (1986) method, assigning each node the mass of the interval delimited by the midpoints of the new grid,

$$m_{i,\ell}^{(r+1)} = \frac{\xi_i^{(\ell,r+1)} + \xi_i^{(\ell+1,r+1)}}{2}, \quad \ell = 1, \dots, N-1.$$

These midpoints induce a partition of the support into N intervals. Each node $\xi_i^{(\ell,r+1)}$ is assigned the probability mass of its surrounding interval, namely

$$\begin{aligned} \left(-\infty, m_{i,1}^{(r+1)}\right) &\text{ for } \ell = 1, \\ \left(m_{i,\ell-1}^{(r+1)}, m_{i,\ell}^{(r+1)}\right) &\text{ for } 1 < \ell < N, \\ \left(m_{i,N-1}^{(r+1)}, +\infty\right) &\text{ for } \ell = N. \end{aligned}$$

For each old state $j = 1, \dots, N$ and new state $\ell = 1, \dots, N$, the entries of the sectoral bridge matrix $\mathbf{\Pi}_i^{(r+1|r)} \in \mathbb{R}^{N \times N}$ are

$$\Pi_{i,j,\ell}^{(r+1|r)} = \begin{cases} \Phi\left(\dfrac{m_{i,1}^{(r+1)} - \rho_i \xi_i^{(j,r)}}{\sigma_i^{(r+1)}}\right) & \ell = 1 \\ \Phi\left(\dfrac{m_{i,\ell}^{(r+1)} - \rho_i \xi_i^{(j,r)}}{\sigma_i^{(r+1)}}\right) - \Phi\left(\dfrac{m_{i,\ell-1}^{(r+1)} - \rho_i \xi_i^{(j,r)}}{\sigma_i^{(r+1)}}\right) & 1 < \ell < N \\ 1 - \Phi\left(\dfrac{m_{i,N-1}^{(r+1)} - \rho_i \xi_i^{(j,r)}}{\sigma_i^{(r+1)}}\right) & \ell = N \end{cases}$$

where $\Phi(\cdot)$ denotes the CDF of the standard normal distribution. By construction, $\sum_{\ell=1}^{N} \Pi_{i,j,\ell}^{(r+1|r)} = 1$ for every j.

Since sectoral shocks are independent ($\nu_{m,t} \perp \nu_{s,t}$), the joint bridge matrix for the full exogenous state space is obtained as the Kronecker product of the sectoral bridge matrices

$$\mathbf{\Pi}_{\tilde{z}}^{(r+1|r)} = \mathbf{\Pi}_{m}^{(r+1|r)} \otimes \mathbf{\Pi}_{s}^{(r+1|r)}, \quad \mathbf{\Pi}_{\tilde{z}}^{(r+1|r)} \in \mathbb{R}^{N^2 \times N^2},$$

consistently with the construction of $\mathbf{\Pi}_{\tilde{z}} = \mathbf{\Pi}_m \otimes \mathbf{\Pi}_s$ as in the constant-volatility case.

Productivity levels in the new regime are recovered by reintroducing the ergodic mean of the new regime, as in (3.5). By Corollary 3.1 the variance inherited from the old regime persists and decays at rate ρ_i^2, so mean neutrality in levels is restored only gradually after a switch, and the simulation reproduces that transitory deviation rather than correcting it. To lighten the notation the regime index has been omitted in the solution stages above, with the understanding that every object refers to the prevailing regime and that consecutive regimes are linked at the switch dates by the bridge $\mathbf{\Pi}_{\tilde{z}}^{(r+1|r)}$.

Finally, repeating to $t = T$ across many Monte Carlo draws generates the distribution of stochastic transition paths implied by the model, capturing both the structural transformation and the dispersion across simulated paths.

References

Autor, D. H., and D. Dorn (2013): "The Growth of Low-Skill Service Jobs and the Polarization of the US Labor Market," *American Economic Review*, 103(5), 1553–1597.

Boppart, T. (2014): "Structural Change and the Kaldor Facts in a Growth Model with Relative Price Effects and Non-Gorman Preferences," *Econometrica*, 82(6), 2167–2196.

Caldara, D., and M. Iacoviello (2022): "Measuring Geopolitical Risk," *American Economic Review*, 112(4), 1194–1225.

Carter, S. B., S. S. Gartner, M. R. Haines, A. L. Olmstead, R. Sutch, G. Wright, et al. (2006): *Historical statistics of the United States: millennial edition*, vol. 3. Cambridge University Press New York.

Cossu, F., A. Moro, and M. P. Rendall (2026): "Training time, robots and technological unemployment," *Journal of Economic Behavior & Organization*, 248, 107631.

Gadea, M. D., A. Gómez-Loscos, and G. Pérez-Quirós (2020): "The decline in volatility in the US economy. A historical perspective," *Oxford Economic Papers*, 72(1), 101–123.

Herrendorf, B., R. Rogerson, and A. Valentinyi (2014): "Growth and Structural Transformation," in *Handbook of Economic Growth*, vol. 2, pp. 855–941. Elsevier.

Kongsamut, P., S. Rebelo, and D. Xie (2001): "Beyond Balanced Growth," *The Review of Economic Studies*, 68(4), 869–882.

Kopecky, K. A., and R. M. Suen (2010): "Finite state Markov-chain approximations to highly persistent processes," *Review of Economic Dynamics*, 13(3), 701–714.

Maddison, A. (2010): "Statistics on world population, GDP and per capita GDP, 1-2008 AD," Groningen Growth and Development Centre.

Ravn, M. O., and H. Uhlig (2002): "On adjusting the Hodrick-Prescott filter for the frequency of observations," *Review of economics and statistics*, 84(2), 371–376.

Rouwenhorst, K. G. (1995): "Asset pricing implications of equilibrium business cycle models," *Frontiers of business cycle research*, 1, 294–330.

Stokey, N. L., R. E. Lucas, and E. C. Prescott (1989): *Recursive Methods in Economic Dynamics*. Harvard University Press.

Tauchen, G. (1986): "Finite state markov-chain approximations to univariate and vector autoregressions," *Economics letters*, 20(2), 177–181.